Stationary measure $\mu = (\mu_1, \dots, \mu_n)$ if MC has $n$ states : $\mu = \mu P$

after transiting for one step remains the same

eigvector corresponding to eigval 1 of $P$

Such stat meas always exists as long as $\{X_n\}$ has a recurrent state $x$ (Kolmogorov's cycle trick)

$$\rho_x(y) \triangleq \sum_{n=0}^{\infty} \mathbb{P}_x(X_n = y, T_x > n)$$


Stat dist: a normalized version of stat meas. $\mu$

$$\pi_i = \frac{\mu_i}{\sum_{j=1}^{n} \mu_j}$$

such that components sum up to 1 and are non-negative.

For $\{X_n\}$ irreducible and recurrent, $\exists\, \mu \geq 0$, $\mu = \mu P$

stat dist exists $\Longleftrightarrow$ $\sum_{j=1}^{n} \mu_j < \infty$

$\Updownarrow$

$\{X_n\}$ positive recurrent

$$\pi_i = \frac{\mu_i}{\sum_{j=1}^{n} \mu_j}$$

$\Updownarrow$

$$\pi_i = \frac{1}{\mathbb{E}_i T_i}$$

($\mathbb{E}_i T_i$ is mean recurrence time for state $i$)

So: positive recur and the existence of $\pi$ basically the same!

Remark: By def of $\pi$,

$$\forall i \in S, \ \pi_i = \sum_{j=1}^{n} \pi_j P_{ji}$$

$$\frac{1}{\mathbb{E}_i T_i} = \sum_{j=1}^{n} P_{ji} \frac{1}{\mathbb{E}_j T_j}$$

<u>Ergodic Thm</u> : relationship between limiting dist and stat dist $\pi$

limiting dist : if $X_0 \sim \mu_0$, $\lim_{n\to\infty} \mu_0 \cdot P^n$ is limiting dist ( dist of $X_n$ as $n\to\infty$)

If $\{X_n\}$ is ergodic ( irred, pos-recurrent, <u>aperiodic</u>), for any initial dist of $X_0$, limiting dist is stat dist!

aperiodicity is required!

Counter eg: $S=\{0,1\}$, $X_0=0$, $P=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$0 \to 1 \to 0 \to 1 \to 0 \to 1 - - -$

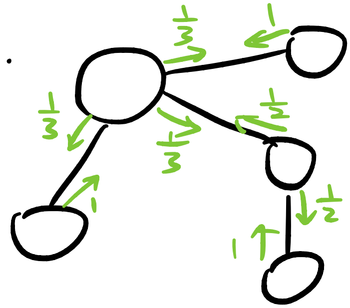
limiting dist does not exist!

(6.4.6)

e.g.: RW on graph, move to neighbor with equal prob

each vertex $v$ has degree $d_v$, check the stat dist is

$$\pi_v = \frac{d_v}{\gamma}, \text{ where } \gamma \triangleq \sum_{u\in V} d_u.$$




Pf:

①: $\forall v\in V,\ \pi_v \geq 0$

②: $\sum_{v\in V} \pi_v = \frac{\sum_{v\in V} d_v}{\gamma} = 1$

③: Check $\pi = \pi P$, $\forall v\in V$, $\pi_v = \sum_{u\in V} \pi_u P_{uv}$

$$RHS = \sum_{u\in V} \frac{d_u}{\gamma}\cdot\frac{1}{d_u}\cdot I_{\underbrace{\{u\sim v\}}_{\text{edge }(u,v)\text{ exists}}} = \frac{1}{\gamma}\sum_{u\in V} I_{\{u\sim v\}}$$

$$= \frac{d_v}{\gamma} = \pi_v \quad \checkmark$$

e.g.: (6.4.8) At time $n$, $Y_n$ particles enter, $Y_n \overset{i.i.d.}{\sim} \mathcal{P}(\lambda)$, lifetime of particles i.i.d. $\sim G(p)$, $X_n \triangleq \#$ of particles in the system at time $n$. Show $\{X_n\}$ Markov, and find $\pi$.

Pf: From time $n-1$ to $n$, $Y_n$ particles added, some of $X_{n-1}$ particles vanish.

$G(p)$ memoryless, no matter how long a particle has lived, the probability of vanishing is always $p$

For $Z \sim G(p)$, $\forall n \in \mathbb{N}$,

$$\mathbb{P}(Z \geqslant n+1 \mid Z \geqslant n) = \frac{\mathbb{P}(Z \geqslant n+1)}{\mathbb{P}(Z \geqslant n)} = 1-p$$

☆: Memoryless property is the key for $\{X_n\}$ to be Markov $\begin{cases} \text{discrete} & \text{— } G(p) \\ \text{cts} & \text{— } \mathcal{E}(\lambda) \end{cases}$

Among $X_{n-1}$ particles, there are $B(X_{n-1}, 1-p)$ particles alive.

$$X_n = \sum_{i=1}^{X_{n-1}} \xi_i^{(n)} + Y_n \ , \ \{\xi_i^{(n)}\} \sim B(1, 1-p) \text{ i.i.d.}$$

with $\{Y_n\}$, $\{X_n\}$, $\{\xi_i^{(n)}\}$ indep.

Then

$$\mathbb{P}(X_n = x_n \mid X_0 = x_0, \dots, X_{n-1} = x_{n-1}) =$$

$$\mathbb{P}\Big(\sum_{i=1}^{x_{n-1}} \xi_i + Y_n = x_n \,\Big|\, X_0 = x_0, \dots, X_{n-1} = x_{n-1}\Big)$$

indep

$$= \mathbb{P}\Big(\sum_{i=1}^{x_{n-1}} \xi_i + Y_n = x_n\Big) = \mathbb{P}(X_n = x_n \mid X_{n-1} = x_{n-1}) \checkmark$$

similar

Let $\pi$ be stat dist so that

$$\forall i, \ \pi_i = \sum_j \pi_j \cdot P_{ji},$$

$$P_{ji} = \mathbb{P}(X_n = i \mid X_{n-1} = j) = \mathbb{P}\Big(\underbrace{\sum_{k=1}^{j} \xi_k^{(n)}}_{B(j,1-p)} + \underbrace{Y_n}_{\mathcal{P}(\lambda)} = i\Big)$$

$$= \begin{cases} \displaystyle\sum_{k=0}^{i} \binom{j}{k} (1-p)^k p^{j-k} \cdot \frac{\lambda^{i-k}}{(i-k)!} e^{-\lambda} & \text{if } i \le j \\ \displaystyle\sum_{k=0}^{j} \binom{j}{k} (1-p)^k p^{j-k} \frac{\lambda^{i-k}}{(i-k)!} e^{-\lambda} & \text{if } i > j \end{cases}$$

So: $$\forall i, \ \pi_i = \sum_{j=0}^{\infty} \pi_j \cdot \sum_{k=0}^{i \wedge j} \binom{j}{k} (1-p)^k p^{j-k} \frac{\lambda^{i-k}}{(i-k)!} e^{-\lambda} \quad (*)$$

---

It's very hard to compute $\pi$ by def due to the structure of $\sum_{k=1}^{j} \xi_k^{(n)} + Y_n$ (sum of indep r.v.)

Hence, think about using g.f. instead.

$$G_{X_{n+1}}(s) = \mathbb{E}s^{X_{n+1}} = \mathbb{E}s^{\sum_{i=1}^{X_n} \rho_i^{(n+1)} + Y_{n+1}}$$

$$= \mathbb{E}s^{Y_{n+1}} \cdot \mathbb{E}\left[\mathbb{E}\left(s^{\sum_{i=1}^{X_n} \rho_i^{(n+1)}} \middle| X_n\right)\right] \qquad \left(\mathbb{E}s^{\rho_1^{(n+1)}}\right)^{X_n} = \left[p+(1-p)s\right]^{X_n}$$

$$= e^{\lambda(s-1)} \cdot \mathbb{E}\left[p+(1-p)s\right]^{X_n}$$

$$= e^{\lambda(s-1)} \cdot G_{X_n}\left(p+(1-p)s\right)$$

If $X_0 \sim \pi$, $\forall n$, $X_n \sim \pi$, so $\forall n$, $G_{X_n}(s) = G_\pi(s)$

$$G_\pi(s) = e^{\lambda(s-1)} \cdot G_\pi(p+(1-p)s)$$

$$\Downarrow$$

$$\log G_\pi(s) = \lambda(s-1) + \log G_\pi(p+(1-p)s)$$

$$\Downarrow$$

$$\log G_\pi(s) = \frac{\lambda}{p}(s-1)$$

$$\Downarrow$$

$$G_\pi(s) = e^{\frac{\lambda}{p}(s-1)},$$

$\boxed{\pi = \mathcal{P}\left(\frac{\lambda}{p}\right)}$ is stat dist.

Check eqn $(*)$: $\pi_i = \frac{\left(\frac{\lambda}{p}\right)^i}{i!} e^{-\frac{\lambda}{p}}$

$$RHS = \sum_{j=0}^{\infty} \left(\frac{\lambda}{p}\right)^j \frac{1}{j!} e^{-\frac{\lambda}{p}} \cdot \sum_{k=0}^{i \wedge j} \frac{j!}{k!(j-k)!} (1-p)^k p^{j-k} \frac{\lambda^{i-k}}{(i-k)!} e^{-\lambda}$$

$$\overset{\text{Fubini}}{=} e^{-\frac{\lambda}{p}} \cdot \cancel{e^{-\lambda}} \cdot \lambda^{i} \cdot \sum_{k=0}^{i} (1-p)^{k} \cdot (p\lambda)^{-k} \cdot \frac{1}{(i-k)!} \cdot \underbrace{\sum_{j=k}^{\infty} \left(\frac{\lambda}{p}\right)^{j} \cdot \frac{1}{j!} \binom{j}{k} p^{j}}_{\sum_{j=k}^{\infty} \binom{j}{k} \frac{\lambda^{j}}{j!} = \frac{\lambda^{k}}{k!} \sum_{\ell=0}^{\infty} \frac{\lambda^{\ell}}{\ell!} = \frac{\lambda^{k}}{k!} \cdot \cancel{e^{\lambda}}}$$

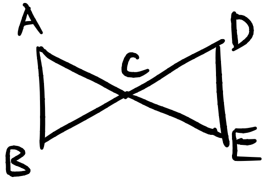
$$= e^{-\frac{\lambda}{p}} \lambda^{i} \cdot \sum_{k=0}^{i} \left(\frac{1-p}{p}\right)^{k} \cdot \underbrace{\frac{1}{k!(i-k)!}}_{= \frac{1}{i!}\binom{i}{k}}$$

$$= e^{-\frac{\lambda}{p}} \cdot \lambda^{i} \cdot \frac{1}{i!} \underbrace{\sum_{k=0}^{i} \left(\frac{1-p}{p}\right)^{k} \cdot \binom{i}{k}}_{\left(1+\frac{1-p}{p}\right)^{i} = p^{-i}}$$

$$= \left(\frac{\lambda}{p}\right)^{i} \cdot \frac{1}{i!} e^{-\frac{\lambda}{p}} = \pi_i \quad \checkmark$$

We checked that $\pi = \mathcal{P}\left(\frac{\lambda}{p}\right)$ is stat dist by def.

e.g: (6.4.11)




RW on graph, $X_0 = A$

(a): Find $\mathbb{E}_A T_A$, $T_A$ is first hitting time to A except time 0

Pf: Mean recurrence time $\longrightarrow$ stat dist

By symmetricity, $\pi_A = \pi_B = \pi_D = \pi_E$

So: $\pi = (a, a, 1-4a, a, a)$ with $a \in [0, \frac{1}{4}]$

By def, $\pi_C = \pi_A \cdot P_{AC} + \pi_B \cdot P_{BC} + \pi_D \cdot P_{DC} + \pi_E \cdot P_{EC}$

$$1 - 4a = 2a, \quad \boxed{a = \frac{1}{6}}$$

$$\pi = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

$$\mathbb{E}_A T_A = \frac{1}{\pi_A} = \boxed{6}$$

(b): Find expected # of visits to D before returning to A.

Pf: Consider $\rho_A(D) = \sum_{n=0}^{\infty} \mathbb{P}_A(X_n = D, T_A > n)$ as stat meas. (since all states recurrent).

By uniqueness of stat meas for irred pos recurrent Markov chain, $\pi_D = \dfrac{\rho_A(D)}{\sum_s \rho_A(s)}$

$$\sum_s \mu_A(s) = \sum_s \sum_{n=0}^{\infty} \mathbb{P}_A(X_n = s, T_A > n) \overset{\text{Fubini}}{=} \sum_{n=0}^{\infty} \mathbb{P}_A(T_A > n)$$

$$= \mathbb{E}_A T_A$$

So: $\mu_A(D) = \pi_D \cdot \mathbb{E}_A T_A = \frac{1}{6} \cdot 6 = \boxed{1}$

(c): Find expected # of visits to C before returning to A

Pf: Similarly, $\mu_A(C) = \pi_C \cdot \mathbb{E}_A T_A = \frac{1}{3} \cdot 6 = \boxed{2}$

(d): Find expected time of first return to A, given no prior visit to E. (*)

Pf: $\mathbb{E}_A(T_A \mid T_E > T_A)$ (first step decomp)

$$= \sum_s \mathbb{P}_A(X_1 = s \mid T_E > T_A) \cdot \underbrace{\mathbb{E}_A(T_A \mid T_E > T_A, X_1 = s)}_{\overset{\text{Markov}}{=} \begin{cases} 1 & \text{if } s = A \\ 1 + \mathbb{E}_s(T_A \mid T_E > T_A) & \text{if } s \neq A,\ s \neq E \end{cases}}$$

What are the probabilities $\mathbb{P}_A(X_1 = s \mid T_E > T_A)$?

$$= \frac{\mathbb{P}_A(X_1 = s, T_E > T_A)}{\mathbb{P}_A(T_E > T_A)} = \frac{\mathbb{P}_A(X_1 = s) \cdot \underbrace{\mathbb{P}_A(T_E > T_A \mid X_1 = s)}_{\overset{\text{Markov}}{=} \begin{cases} 1 & \text{if } s = A \\ \mathbb{P}_s(T_E > T_A) & \text{if } s \neq A,\ s \neq E \end{cases}}}{\mathbb{P}_A(T_E > T_A)}$$

Everything reduces to calculating $\mathbb{P}_s(T_E > T_A)$ for $\forall s$.

<u>First step decomposition:</u>

$$\mathbb{P}_A(T_E > T_A) = \sum_s \mathbb{P}_A(X_1 = s) \cdot \mathbb{P}_A(T_E > T_A \mid X_1 = s)$$
$$\overset{\text{Markov}}{=} P_{AA} \cdot 1 + P_{AB} \cdot \mathbb{P}_B(T_E > T_A) + P_{AC} \cdot \mathbb{P}_C(T_E > T_A) + P_{AD} \cdot \mathbb{P}_D(T_E > T_A)$$

A similar expansion holds for $\mathbb{P}_B(T_E > T_A)$, ---.

So:

$$\begin{cases} \mathbb{P}_A(T_E > T_A) = \frac{1}{2}\mathbb{P}_B(T_E > T_A) + \frac{1}{2}\mathbb{P}_C(T_E > T_A) \\ \mathbb{P}_B(--) = \frac{1}{2} + \frac{1}{2}\mathbb{P}_C(--) \\ \mathbb{P}_C(--) = \frac{1}{4} + \frac{1}{4}\mathbb{P}_B(--) + \frac{1}{4}\mathbb{P}_D(--) \\ \mathbb{P}_D(--) = \frac{1}{2}\mathbb{P}_C(--) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \mathbb{P}_A(T_E > T_A) = \frac{5}{8} \\ \mathbb{P}_B(--) = \frac{3}{4} \\ \mathbb{P}_C(--) = \frac{1}{2} \\ \mathbb{P}_D(--) = \frac{1}{4} \end{cases}$$

So:

$$\begin{cases} \mathbb{P}_A(X_1=A \mid T_E>T_A) = \dfrac{\mathbb{P}_A(X_1=A)\cdot 1}{\mathbb{P}_A(T_E>T_A)} = 0 \\ \mathbb{P}_A(X_1=B \mid T_E>T_A) = \dfrac{\mathbb{P}_A(X_1=B)\cdot \mathbb{P}_B(T_E>T_A)}{\mathbb{P}_A(T_E>T_A)} = \dfrac{3}{5} \\ \mathbb{P}_A(X_1=C \mid - \, -) = \dfrac{2}{5} \end{cases}$$

$$\Downarrow$$

$$\mathbb{E}_A(T_A \mid T_E>T_A) = 1 + \tfrac{3}{5}\cdot \mathbb{E}_B(T_A \mid T_E>T_A) + \tfrac{2}{5}\cdot \mathbb{E}_C(T_A \mid T_E>T_A)$$

What about $\mathbb{E}_B(T_A \mid T_E>T_A)$ and $\mathbb{E}_C(T_A \mid T_E>T_A)$?

Similar Technique! We would need

$$\begin{cases} \mathbb{P}_B(X_1=A \mid T_E>T_A) = \dfrac{\mathbb{P}_B(X_1=A)\cdot 1}{\mathbb{P}_B(T_E>T_A)} = \dfrac{2}{3} \\ \mathbb{P}_B(X_1=C \mid T_E>T_A) = \dfrac{\mathbb{P}_B(X_1=C)\cdot \mathbb{P}_C(T_E>T_A)}{\mathbb{P}_B(T_E>T_A)} = \dfrac{1}{3} \end{cases}$$

and

$$\begin{cases} \mathbb{P}_C(X_1=A \mid T_E>T_A) = \dfrac{\mathbb{P}_C(X_1=A)\cdot 1}{\mathbb{P}_C(T_E>T_A)} = \dfrac{1}{2} \\ \mathbb{P}_C(X_1=B \mid T_E>T_A) = \dfrac{\mathbb{P}_C(X_1=B)\cdot \mathbb{P}_B(T_E>T_A)}{\mathbb{P}_C(T_E>T_A)} = \dfrac{3}{8} \\ \mathbb{P}_C(X_1=D \mid T_E>T_A) = \dfrac{\mathbb{P}_C(X_1=D)\cdot \mathbb{P}_D(T_E>T_A)}{\mathbb{P}_C(T_E>T_A)} = \dfrac{1}{8} \end{cases}$$

and

$$\mathbb{P}_D(X_1=C \mid T_E>T_A) = 1$$

So:

$$\mathbb{E}_B(T_A \mid T_E>T_A) = 1 + \tfrac{1}{3}\,\mathbb{E}_C(T_A \mid T_E>T_A)$$

$$\mathbb{E}_C(T_A \mid T_E>T_A) = 1 + \tfrac{3}{8}\,\mathbb{E}_B(T_A \mid T_E>T_A) + \tfrac{1}{8}\,\mathbb{E}_D(T_A \mid T_E>T_A)$$

$$\mathbb{E}_D(T_A \mid T_E>T_A) = 1 + \mathbb{E}_C(T_A \mid T_E>T_A)$$

Combine four red eqns:

$$\begin{cases} \mathbb{E}_A(T_A \mid T_E > T_A) = \boxed{\frac{14}{5}} \\ \mathbb{E}_B(T_A \mid T_E > T_A) = \frac{5}{3} \\ \mathbb{E}_C(\text{— — -}) = 2 \\ \mathbb{E}_D(\text{— — -}) = 3 \end{cases}$$

(e): Find expected # of visits to D before returning to A, given no prior visit to E.

Pf: Conditional transition prob under $T_E > T_A$ is

$$P' = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & & \frac{3}{5} & \frac{2}{5} & \\ B & \frac{2}{3} & & \frac{1}{3} & \\ C & \frac{1}{2} & \frac{3}{8} & & \frac{1}{8} \\ D & & & 1 & \end{array}$$

from previous problem. The stationary dist of this conditional transition is $\pi' = \left(\frac{10}{28}, \frac{9}{28}, \frac{8}{28}, \frac{1}{28}\right)$

(solve from $\pi' = \pi' P'$)

By same reasoning in (b), answer is

$$\pi'_D \cdot \mathbb{E}_A(T_A \mid T_E > T_A) = \frac{1}{28} \cdot \frac{14}{5} = \boxed{\frac{1}{10}}$$